Symmetries and Noether's theorem

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1 Scalar field action and associated symmetries

A theory of the scalar field, $\Phi(x)$ can be described by the action,

$$I\left[\Phi(x)\right] = \int d^4x \, \mathcal{L},$$

where the Lagrangian $\mathcal{L}$ is a function of the scalar, Φ and it's spacetime derivatives $\partial_\mu \Phi$,

$$\mathcal{L} = \mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right).$$

The integration range is over all space and time.

For the **free** scalar field, the Lagrangian can be taken to be,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \, \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2. \tag{1}$$

This form is dictated by Lorentz invariance i.e. the Lagrangian density **must be Lorentz scalar** and the funny factors of $-\frac{1}{2}$ are there so that the kinetic terms come with the right sign and right factor, e.g. $\frac{1}{2}\dot{\Phi}^2$. m is invariant quantity with the dimensions of mass (or energy). Upon quantizing the system, the parameter, m will turn out to be the mass of the scalar field quanta/particles.

The symmetries/invariances of the scalar field action are:

- Lorentz invariance $x \to x' = \Lambda x$, $\Phi(x) \to \Phi'(x') = \Phi(x)$.
- Translational invariance $x \to x' = x + a$, $\Phi(x) \to \Phi'(x') = \Phi(x)$.

Check:

- Lorentz invariance is rather obvious because most terms in the action is Lorentz invariant, d^4x, m^2, Φ^2. Even the kinetic term, $\partial_\mu \Phi \, \partial^\mu \Phi$, is because the Lorentz index μ is contracted, viz:

$$\partial_\mu \Phi(x) \to \partial'_\mu \Phi'(x') = \Lambda_\mu{}^\nu \, \partial_\nu \Phi(x),$$

$$
\begin{aligned}
\Longrightarrow \partial_\mu \Phi(x)\, \partial^\mu \Phi(x) \to \partial'_\mu \Phi'(x')\partial'^\mu \Phi'(x') &= \Lambda_\mu{}^\nu\, \partial_\nu \Phi(x) \quad \Lambda^\mu{}_\alpha \partial^\alpha \Phi(x) \\
&= (\Lambda_\mu{}^\nu)\ (\Lambda^\mu{}_\alpha)\ \partial_\nu \Phi(x)\, \partial^\alpha \Phi(x) \\
&= \delta^\nu_\alpha\, \partial_\nu \Phi(x)\, \partial^\alpha \Phi(x) \\
&= \partial_\nu \Phi(x)\, \partial^\nu \Phi(x).
\end{aligned}
$$

Thus the action

$$
\begin{aligned}
I\left[\Phi'(x')\right] &= \int d^4x' \left[-\frac{1}{2}\partial'_\mu \Phi'(x')\partial'^\mu \Phi'(x') - \frac{1}{2}m^2 \Phi'(x)^2\right] \\
&= \int d^4x \left[-\frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x) - \frac{1}{2}m^2 \Phi(x)^2\right] \\
&= I\left[\Phi(x)\right],
\end{aligned}
$$

remains invariant.

- Translation invariance is also obvious because the action integral being defined over all space and time i.e ranges of integration being $(-\infty, \infty)$, is independent of the origin of coordinates and there is no *explicit* dependence on the coordinates, x. What about $\partial_\mu \Phi$. This term seems to care about the spacetime coordinate through the derivative, $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Actually even this derivative is independent of the shift in origin because under a shift of origin of coordinates,

$$x \to x' = x + a,$$

Conversely,

$$x = x' - a$$

the derivative transforms as

$$
\begin{aligned}
\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}, \\
&= \frac{\partial\left(x'^\nu - a^\nu\right)}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\
&= \frac{\partial x'^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\
&= \partial^\nu_\mu \frac{\partial}{\partial x^\nu} \\
&= \frac{\partial}{\partial x^\mu}.
\end{aligned}
$$

So the derivative remains unchanged. (Here since a is a constant its derivative vanishes, and we get $\frac{\partial(x'^\nu - a^\nu)}{\partial x'^\mu} = \frac{\partial x'^\nu}{\partial x'^\mu}$).

2 Complex Scalar field theory

A quick generalization of the free real scalar field theory is obtained by complexifying the field. This new complex scalar field theory has a new **internal** symmetry called the global U(1) symmetry which we will consider as well. Since the action must be real, the action is constrained to be the following,

$$I\left[\Phi(x),\Phi^\dagger(x)\right] = \int d^4x\,\left[(\partial_\mu\Phi)^\dagger\,\partial^\mu\Phi - m^2\Phi^\dagger\Phi\right]. \tag{2}$$

Here $\Phi^\dagger$ is the complex conjugate of Φ. The classical equation of motion for the complex field is same as that of the real scalar field, i.e. the Klein-Gordon equation,

$$\left(\partial^2+m^2\right)\Phi = \left(\partial^2+m^2\right)\Phi^\dagger = 0.$$

Upon rewriting this complex scalar field into it's real and imaginary components,

$$\Phi = \frac{\phi_1 + i\,\phi_2}{\sqrt{2}},\, \Phi^\dagger = \frac{\phi_1 - i\,\phi_2}{\sqrt{2}},$$

we find out that the complex scalar field theory is a theory of two **independent (non-interacting)** real scalar field theory. This is because on splitting the field into its real and imaginary components, the action splits into two pieces as well,

$$\begin{aligned} I\left[\Phi(x),\Phi^\dagger(x)\right] &= \int d^4x\,\left[(\partial_\mu\Phi)^\dagger\,\partial^\mu\Phi - m^2\Phi^\dagger\Phi\right], \\ &= \int d^4x\,\left(\frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2\right) + \int d^4x\,\left(\frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2\right). \end{aligned}$$

2.1 Global $U(1)$ Symmetry of the complex field theory

One can easily check that the complex scalar field theory action Eq. (2) is invariant under multiplication by a **constant** complex phase factor $e^{i\,\alpha}$,

$$\begin{aligned} \Phi \to \Phi' &= e^{-i\,\alpha}\Phi, \\ \Phi^\dagger \to \Phi'^\dagger &= e^{i\,\alpha}\Phi^\dagger. \end{aligned} \tag{3}$$

The phase,α is necessarily a real number. Since a complex phase is unitary i.e. the complex conjugation is also the inverse,

$$\left(e^{-i\,\alpha}\right)^\dagger = \left(e^{-i\,\alpha}\right)^{-1},$$

such phases are also called $U(1)$ factors (U stands for Unitary matrix and since a number is a 1×1 matrix, $U(1)$ is unitary matrix of size 1×1). Since this symmetry transformation does not touch spacetime but only changes the fields, such a symmetry is called an **internal symmetry**. Also note that since α is a constant i.e. not a function of spacetime, it is a **global** symmetry (**global = same everywhere = independent of spacetime location**).

Check: Under the U(1) symmetry Eq. (3), the mass term is obviously invariant,

$$\begin{aligned} \Phi'^{\dagger}\Phi' &= \left(e^{i\,\alpha}\Phi^{\dagger}\right)\,\left(e^{-i\,\alpha}\Phi\right) \\ &= \Phi^{\dagger}\Phi \end{aligned}$$

and this is true whether α is a constant or a function of spacetime i.e. $\alpha(x)$. Now let's look at the kinetic term,

$$\begin{aligned} \left(\partial_{\mu}\Phi^{\dagger}\right)\left(\partial^{\mu}\Phi\right) \to \left(\partial_{\mu}\Phi'^{\dagger}\right)\left(\partial^{\mu}\Phi'\right) &= \partial_{\mu}\left(e^{i\,\alpha}\Phi^{\dagger}\right)\partial^{\mu}\left(e^{-i\,\alpha}\Phi\right), \\ &= e^{i\,\alpha}\left(\partial_{\mu}\Phi^{\dagger}\right)\; e^{-i\,\alpha}\left(\partial^{\mu}\Phi\right) \\ &= \left(\partial_{\mu}\Phi^{\dagger}\right)\left(\partial^{\mu}\Phi\right). \end{aligned}$$

So this kinetic term in the action is also invariant because α is a constant and the derivative does not act on it. If α was a function of spacetime, $\alpha = \alpha(x)$, the derivative would have acted on it and the term would not be invariant. Incidentally, a spacetime dependent phase $\alpha(x)$ is called a **local $U(1)$ transformation.**

3 Noether's theorem and conserved charges

We will use Noether's theorem to construct the conserved charges for the free scalar system. Recall that Noether's theorem states that conserved charges only appear for *global* symmetries i.e. when the symmetry transformation is same all over spacetime. In the present case, we see that the parameters $\Lambda^{\mu}{}_{\nu}$ and a^{μ} are indeed constants and not functions of spacetime. So Noether's theorem applies in these cases and tells us that there must be conserved charges. Here we obtain the expressions for those charges using the so called "Noether procedure", where we will **temporarily assume** that the symmetry parameters are functions of spacetime, i.e. $\Lambda^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}(x) = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}(x)$ and $a^{\mu} = a^{\mu}(x)$. Then (to lowest order), the change in the action integral should be,

$$\delta I = \int d^4x\; \left(\partial_{\mu}a^{\nu}\right)\; \mathfrak{T}^{\mu}{}_{\nu}$$

or

$$\delta I = \int d^4x\; \left(\partial_{\rho}\omega^{\mu}{}_{\nu}\right)\; M^{\mu}{}_{\nu\rho}.$$

This form is consistent with our expectation that in the special case when a and ω are constants these expressions must vanish as the action is invariant under the global/constant changes. The companion coefficient terms i.e. $T^{\mu}{}_{\nu}$ or $M^{\mu}{}_{\nu\rho}$ which are some functions of scalar field and it's derivatives are the conserved quantities. This can be inferred from the above expressions by a simple integration by parts and abandoning the total derivative term (we can abandon this term under the assumption that the surface term goes to zero at infinity). For example for the translation invariance: s

$$\begin{aligned}\delta I &= \int d^4x \; (\partial_\mu a^\nu(x)) \; \mathfrak{T}^\mu{}_\nu \\ &= \int d^4x \, \partial_\mu \left(a^\nu(x) \, \mathfrak{T}^\mu{}_\nu\right) - \int d^4x \; (\partial_\mu \mathfrak{T}^\mu{}_\nu) \; a^\nu(x) \\ &= -\int d^4x \; (\partial_\mu \mathfrak{T}^\mu{}_\nu) \; a^\nu(x).\end{aligned}$$

Now when we go to the global case, i.e. a^ν becomes a constant, it can be pulled out of the integral and we have the expression,

$$\delta I = -a^\nu \int d^4x \; (\partial_\mu \mathfrak{T}^\mu{}_\nu) \, .$$

Since the first order variation of the action is zero, i.e. when the equation of motion is valid i.e. for this global a^ν is zero, the integrand must vanish. The vanishing of the integrand is nothing but the continuity equation,

$$\partial_\mu \mathfrak{T}^\mu{}_\nu = 0,$$

in other words, $T^\mu{}_\nu$ is a conserved current. In the following we use the Noether procedure to extract the conserved quantities, $M^{\mu\nu\rho}$ and $\mathfrak{T}^\mu{}_\nu^{\mu\nu}$ explicitly in terms of Φ and it's derivatives, $\partial_\mu \Phi$.

3.1 Translation symmetry and the conserved "Stress-Energy-Momentum" tensor

The conserved quantity corresponding to translation/shift symmetry of the scalar field system is called the *Stress-Energy-Momentum* tensor and is denoted by $T^{\mu\nu}$. We use the Noether procedure to extract this as follows. First step is to turn the shift parameter to be not global, but instead a function of spacetime,

$$a^\mu = a^\mu(x).$$

So we have new coordinates,

$$x \to x'^\mu = x^\mu + a^\mu(x).$$

The Jacobian matrix components for the change of variables, $x \to x'$ are,

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu a^\mu,$$

i.e. it is a sum of the identity matrix and a small change, $\partial_\nu a^\mu$. Hence to first order of the shift, the Jacobian determinant is,

$$\begin{aligned}|J| &= 1 + \mathrm{trace}(\partial_\nu a^\mu) \\ &= 1 + \partial_\rho a^\rho. \qquad (4)\end{aligned}$$

The derivatives transform like,

$$
\begin{aligned}
\partial_\mu \to \partial'_\mu &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \\
&= \left(\delta^\nu_\mu - \partial_\mu a^\nu\right)\,\partial_\nu \\
&= \partial_\mu - \partial_\mu a^\nu\,\partial_\nu. \qquad (5)
\end{aligned}
$$

Now let's look at the change in Φ and it's derivatives, $\partial_\mu \Phi$. We have,

$$\Phi'(x') = \Phi(x),$$

while,

$$
\begin{aligned}
\partial'_\mu \Phi'(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \Phi(x) \\
&= \partial_\mu \Phi(x) - \partial_\mu a^\nu\,\partial_\nu \Phi(x). \qquad (6)
\end{aligned}
$$

Hence the action after the translation becomes,

$$
\begin{aligned}
I\left[\Phi'(x')\right] &= \int d^4x'\,\mathcal{L}\left(\Phi'(x'), \partial'_\mu \Phi'(x')\right) \\
&= \int d^4x\,|J|\,\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x) - \partial_\mu a^\nu\,\partial_\nu \Phi(x)\right) \\
&= \int d^4x\,\left(1 + \partial_\rho a^\rho\right)\left[\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\mu a^\nu\,\partial_\nu \Phi(x) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi(x)\right.} + O(a^2)\right] \\
&= \int d^4x\,\left[\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) + \partial_\rho a^\rho\,\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\mu a^\nu\,\partial_\nu \Phi(x) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi(x)\right.} + O(a^2)\right] \\
&= I\left[\Phi(x)\right] + \int d^4x\,\left[\partial_\rho a^\rho\,\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\mu a^\nu\,\partial_\nu \Phi(x) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi(x)\right.}\right] + O(a^2),
\end{aligned}
$$

This implies, to first order in a

$$
\begin{aligned}
\delta I \equiv I\left[\Phi'(x')\right] - I\left[\Phi(x)\right] &= \int d^4x\,\left[\partial_\rho a^\rho\,\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\mu a^\nu\,\partial_\nu \Phi(x) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi(x)\right.}\right] \\
&= \int d^4x\,\partial_\mu a^\nu\left[\delta^\mu_\nu \mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\nu \Phi(x) \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi(x)\right)}\right] \\
&= \int d^4x\,\partial_\mu a^\nu\,\mathfrak{T}^\mu{}_\nu,
\end{aligned}
$$

where we have identified the conserved current,

$$\mathfrak{T}^{\mu}{}_{\nu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi(x)\right.}\partial_{\nu}\Phi(x) - \delta^{\mu}_{\nu}\mathcal{L}.$$

Raising the ν-index we arrive at the expression of the *canonical* stress-energy-momentum tensor,

$$\begin{aligned} \mathfrak{T}^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi(x)\right.}\partial^{\nu}\Phi(x) - \eta^{\mu\nu}\mathcal{L}, \\ &= \left(\partial^{\mu}\Phi\right)\left(\partial^{\nu}\Phi\right) - \eta^{\mu\nu}\mathcal{L}. \end{aligned} \tag{7}$$

We note couple of things about this canonical stress tensor,

- One can go ahead and check that charges corresponding to the current is nothing but the four-momentum, P^{μ}, i.e.

  $$P^{\nu} = \int d^3\mathbf{x}\, \mathfrak{T}^{0\nu}.$$

  Let's evaluate the 00-component i.e. *energy density*, $\mathfrak{T}^{00}$,

  $$\begin{aligned} \mathfrak{T}^{00} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_{0}\Phi(x)\right.}\partial^{0}\Phi(x) - \eta^{00}\mathcal{L} \\ &= \left(\partial^{0}\Phi(x)\right)^2 - \mathcal{L}, \\ &= \left(\partial^{0}\Phi(x)\right)^2 - \frac{1}{2}\partial_{\mu}\Phi\,\partial^{\mu}\Phi + \frac{1}{2}m^2\Phi^2, \\ &= \left(\partial_0\Phi\right)^2 - \frac{1}{2}\left(\partial_0\Phi\right)^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\Phi\right)^2 + \frac{1}{2}m^2\Phi^2, \\ &= \frac{1}{2}\left(\partial_0\Phi\right)^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\Phi\right)^2 + \frac{1}{2}m^2\Phi^2. \end{aligned}$$

  This expression being a sum of squares is manifestly positive. This is reassuring because we would want a free system to have energy positive semi-definite.

- This tensor is symmetric between the indices, μ, ν. This is only true for scalar fields. For the Maxwell field, we will see that the corresponding stress tensor will *not* be symmetric.

- Note that the stress-energy-momentum is non-unique to some extent. One can always add a term like, $\partial_{\lambda}B^{\lambda\mu\nu}$ where B is a tensor that has the following antisymmetric properties,

  $$B^{\lambda\mu\nu} = -B^{\mu\lambda\nu}.$$

The new quantity[1],

$$T^{\mu\nu} = \mathfrak{T}^{\mu\nu} + \partial_\lambda B^{\lambda\mu\nu}$$

is also conserved,

$$\partial_\mu T^{\mu\nu} = \partial_\mu \mathfrak{T}^{\mu\nu} + \partial_\mu \partial_\lambda B^{\lambda\mu\nu} = 0.$$

For the Maxwell field, one can exploit this ambiguity to define a stress tensor which is symmetric in the indices, μ and ν,

$$T^{\mu\nu} = T^{\nu\mu}.$$

Before we do that we need to first obtain the expression for the charges conserved as a result of Lorentz invariance.

3.2 Conserved charges for Lorentz symmetry

We use the Noether method to extract the charges conserved as a result of Lorentz symmetry of the scalar field action. The first step is to make the Lorentz transformation spacetime dependent,

$$\Lambda^\mu{}_\nu \to \Lambda^\mu{}_\nu(x)$$

In the infinitesimal form we have,

$$\Lambda^\mu{}_\nu(x) = \delta^\mu_\nu + \omega^\mu{}_\nu(x).$$

So the Jacobian matrix is,

$$J^\mu{}_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \omega^\mu{}_\nu(x) + \left(\partial_\nu \omega^\mu{}_\rho(x)\right)\, x^\rho$$

and the Jacobian determinant is,

$$\begin{aligned} |J| &= 1 + \omega^\rho{}_\rho + x^\rho \partial_\sigma \omega^\sigma{}_\rho \\ &= 1 + x^\rho \partial_\sigma \omega^\sigma{}_\rho. \end{aligned} \quad (8)$$

Under this the scalar transforms like,

$$\Phi(x) \to \Phi'(x') = \Phi(x),$$

while the derivatives transform to,

$$\begin{aligned} \partial_\mu \Phi(x) \to \partial'_\mu \Phi'(x') &= \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \Phi(x) \\ &= \partial_\mu \Phi(x) + \omega_\mu{}^\nu\, \partial_\nu \Phi(x) + \left(\partial_\mu \omega_\rho{}^\nu\right)\, x^\rho \partial_\nu \Phi(x). \end{aligned} \quad (9)$$

[1]For those who are interested there is a special "symmetrizing improvement term", $B^{\lambda\mu\nu}$ is called the Belinfante-Rosenfield term after the two people who independently arrived at the expression. For the Maxwell field we will see that the variation of action under Lorentz transformation automatically gives us the improved symmetric stress tensor.

The corresponding change in the action integral is,

$$
\begin{aligned}
I\left[\Phi'(x')\right] &= \int d^4x'\,\mathcal{L}\left(\Phi'(x'),\partial'_\mu\Phi'(x')\right)\\
&= \int d^4x\ \left(1+x^\rho\partial_\sigma\omega^\sigma{}_\rho\right)\ \mathcal{L}\left(\Phi(x),\partial_\mu\Phi(x)+\omega_\mu{}^\nu\ \partial_\nu\Phi(x)+\left(\partial_\mu\omega_\rho{}^\nu\right)\ x^\rho\partial_\nu\Phi(x)\right)\\
&= \int d^4x\ \left(1+x^\rho\partial_\sigma\omega^\sigma{}_\rho\right)\ \Bigg[\mathcal{L}\left(\Phi(x),\partial_\mu\Phi(x)\right)+\omega_\mu{}^\nu\partial_\nu\Phi(x)\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\Phi(x))}\\
&\qquad\qquad +\left(\partial_\mu\omega_\rho{}^\nu\right)\ x^\rho\partial_\nu\Phi(x)\ \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\Phi(x))}+O(\omega^2)\Bigg]\\
&= \int d^4x\,\mathcal{L}\left(\Phi(x),\partial_\mu\Phi(x)\right)+\int d^4x\left(x^\rho\partial_\sigma\omega^\sigma{}_\rho\,\mathcal{L}+\partial_\mu\omega_\rho{}^\nu\ x^\rho\partial_\nu\Phi\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)+\ldots+O(\omega^2).
\end{aligned}
$$

where in the last line steps we have only bothered to show the terms which contain derivatives of ω because the non-derivative terms integrate out to zero. Thus the first order variation in the action,

$$
\begin{aligned}
\delta I\equiv I\left[\Phi'(x')\right]-I\left[\Phi(x)\right] &= \int d^4x\ \left(x^\rho\partial_\sigma\omega^\sigma{}_\rho\,\mathcal{L}+\partial_\mu\omega_\rho{}^\nu\ x^\rho\partial_\nu\Phi\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\\
&= \int d^4x\ \partial_\mu\omega_{\nu\rho}\,x^\rho\left(\eta^{\mu\nu}\mathcal{L}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi\right)\\
&= \int d^4x\ \partial_\rho\omega_{\mu\nu}\,x^\nu\mathfrak{T}^{\rho\mu}\\
&= \int d^4x\ \partial_\rho\omega_{\mu\nu}\ \left(x^\nu\mathfrak{T}^{\rho\mu}-x^\mu\mathfrak{T}^{\rho\nu}\right).
\end{aligned}
$$

Thus we have a set of conserved currents for Lorentz transformations,

$$
\partial_\rho M^{\mu\nu\rho}=0,\qquad M^{\mu\nu\rho}\equiv x^\mu\mathfrak{T}^{\rho\nu}-x^\nu\mathfrak{T}^{\rho\mu}. \tag{10}
$$

This object is antisymmetric in the first pair of indices, i.e. μ,ν while it is symmetric in the exchange of the third index, ρ with any one of the first pair of indices i.e. μ or ν .

One can check that the charges i.e., $L^{\mu\nu}\equiv\int d^3\mathbf{x}\,M^{\mu\nu 0}=\int d^3\mathbf{x}\ \left(x^\mu\mathfrak{T}^{0\nu}-x^\nu\mathfrak{T}^{0\mu}\right)$ are simply the components of the angular momentum and the boost generators.

3.3 Global $U(1)$ symmetry of complex scalar field

In this case, the first order change in the field and its complex conjugate is,

$$
\begin{aligned}
\Phi\to\Phi' &= e^{-i\,\alpha}\Phi\\
&= (1-i\alpha+O(\alpha^2))\Phi\\
&\approx \Phi-i\alpha\Phi,
\end{aligned}
$$

while the complex conjugate field (o first order in α) changes to,

$$\Phi'^{\dagger} \approx \Phi^{\dagger} + i\alpha\Phi^{\dagger}.$$

At this point we **temporarily assume**, α is a function of spacetime,

$$\alpha = \alpha(x).$$

This implies the change in the derivative of the field,

$$\begin{aligned} \partial^{\mu}\Phi \to \partial^{\mu}\Phi' &= \partial^{\mu}\left(\Phi - i\alpha(x)\,\Phi\right) \\ &= \partial^{\mu}\Phi - i\left(\partial^{\mu}\alpha\right)\Phi - i\alpha(x)\,\partial^{\mu}\Phi, \end{aligned}$$

and the derivative of the complex conjugate,

$$\begin{aligned} \partial_{\mu}\Phi^{\dagger} \to \partial_{\mu}\Phi'^{\dagger} &= \partial_{\mu}\left(\Phi^{\dagger} + i\alpha(x)\Phi^{\dagger}\right) \\ &= \partial_{\mu}\Phi^{\dagger} + i\left(\partial_{\mu}\alpha\right)\Phi^{\dagger} + i\alpha(x)\,\partial_{\mu}\Phi^{\dagger}. \end{aligned}$$

The action for the transformed fields is,

$$\begin{aligned} I\left[\Phi', \Phi'^{\dagger}\right] &= \int d^4x\,\left[\left(\partial_{\mu}\Phi'\right)^{\dagger}\partial^{\mu}\Phi' - m^2\Phi'^{\dagger}\Phi'\right] \\ &= \int d^4x\,\left[\left(\partial_{\mu}\Phi^{\dagger} + i\left(\partial_{\mu}\alpha\right)\Phi^{\dagger} + i\alpha(x)\,\partial_{\mu}\Phi^{\dagger}\right)\left(\partial^{\mu}\Phi - i\left(\partial^{\mu}\alpha\right)\Phi - i\alpha(x)\,\partial^{\mu}\Phi\right)\right. \\ &\qquad \left. - m^2\left(\Phi^{\dagger} + i\alpha\Phi^{\dagger}\right)\left(\Phi - i\alpha\Phi\right)\right], \\ &= \int d^4x\,\left[\left(\partial_{\mu}\Phi\right)^{\dagger}\partial^{\mu}\Phi - m^2\Phi^{\dagger}\Phi + i\partial_{\mu}\alpha\left(\Phi^{\dagger}\partial^{\mu}\Phi - \Phi\partial_{\mu}\Phi^{\dagger}\right)\right] \\ &= I\left[\Phi, \Phi^{\dagger}\right] + \int d^4x\,\partial_{\mu}\alpha\, i\,\left(\Phi^{\dagger}\partial^{\mu}\Phi - \Phi\partial_{\mu}\Phi^{\dagger}\right). \end{aligned}$$

So the first order change in the action is,

$$\begin{aligned} \delta I &= I\left[\Phi', \Phi'^{\dagger}\right] - I\left[\Phi, \Phi^{\dagger}\right] \\ &= \int d^4x\,\partial_{\mu}\alpha\, i\,\left(\Phi^{\dagger}\partial^{\mu}\Phi - \Phi\partial_{\mu}\Phi^{\dagger}\right). \end{aligned}$$

From this expression we can identify the conserved current corresponding to the global U(1) symmetry,

$$j^{\mu} = i\,\left(\Phi^{\dagger}\partial^{\mu}\Phi - \Phi\partial_{\mu}\Phi^{\dagger}\right).$$

One can easily check that is conserved on-shell (on-shell means the classical equation of motion holds),

$$\begin{aligned} \partial_{\mu}j^{\mu} &= i\left(\Phi^{\dagger}\partial^2\Phi - \Phi\partial^2\Phi^{\dagger}\right) \\ &= i\left(-\Phi^{\dagger}m^2\Phi + \Phi m^2\Phi^{\dagger}\right) \\ &= 0. \end{aligned}$$

Here we have used the equation of motion,

$$\left(\partial^2 + m^2\right)\Phi = \left(\partial^2 + m^2\right)\Phi^\dagger = 0.$$

4 Free Maxwell Field

The action for the free (i.e. sourceless) Maxwell gauge fields is,

$$I = \int d^4x\, \mathcal{L}(A_\mu, \partial_\mu A_\nu), \quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Again, the action is symmetric under translations,

$$\begin{aligned} x \to x' &= x + a, \\ A_\mu(x) \to A'_\mu(x') &= A_\mu(x). \end{aligned}$$

since the integration range is all over spacetime and there is no explicit dependence on x (only implicit dependence on x via the fields). As a result there exists a conserved current, $\mathfrak{T}^{\mu\nu}$, which can be extracted thru the Noether procedure,

$$\begin{aligned} \mathfrak{T}^{\mu\nu} &= \eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\rho\right)}\partial^\nu A_\rho, \\ &= -\frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\rho}\partial^\nu A_\rho. \end{aligned} \tag{11}$$

Evidently this is not symmetric under $\mu \leftrightarrow \nu$.

Let's evaluate the energy density component, $\mathfrak{T}^{00}$,

$$\begin{aligned} \mathfrak{T}^{00} &= -\frac{1}{4}\eta^{00}F_{\rho\sigma}F^{\rho\sigma} + F^{0\rho}\partial^0 A_\rho \\ &= \frac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) + F^{0i}\partial^0 A_i \\ &= -\frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \mathbf{E}\cdot\dot{\mathbf{A}}, \\ &= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \mathbf{E}\cdot\boldsymbol{\nabla}\varphi \\ &= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \boldsymbol{\nabla}\cdot(\mathbf{E}\varphi) - \varphi\boldsymbol{\nabla}\cdot\mathbf{E}. \end{aligned}$$

The last two terms don't contribute as one is a surface term and the other is proportional to charge density via the Maxwell equation, $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho$. So the energy density is,

$$\mathfrak{T}^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right).$$

This is exactly what we obtain in electrodynamics through other means (Newton's law).

What about the angular momentum and boost generators? Here we need to remind ourselves that the Maxwell field carries spin-1 hence the angular momentum must get a contribution from internal spin in addition to orbital term and hence would look different from the scalar example (10). One can go thru the similar steps, and obtain,

$$\begin{aligned}\delta I &= \frac{1}{2}\int d^4x\,\partial_\mu\omega_{\nu\rho}\left[-\left(x^\nu F^{\mu\alpha}F^\rho{}_\alpha - x^\rho F^{\mu\alpha}F^\nu{}_\alpha\right) + \left(\eta^{\mu\nu}x^\rho\mathcal{L} - \eta^{\mu\rho}x^\nu\mathcal{L}\right)\right] \\ &= \frac{1}{2}\int d^4x\,\partial_\mu\omega_{\nu\rho}\left[x^\rho\left(\eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}F^\nu{}_\alpha\right) - \nu\leftrightarrow\rho\,\right].\end{aligned}$$

So this last line helps us identify a conserved current, $M^{\rho\nu\mu}$which are the Lorentz generators

$$\partial_\mu M^{\rho\nu\mu} = 0,$$

So we see that the Lorentz generators for the Maxwell field are can be expressed as,

$$M^{\mu\nu\rho} \equiv x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho},$$

in terms of a *symmetric* tensor, T

$$\begin{aligned}T^{\mu\nu} &\equiv \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}F^\nu{}_\alpha \\ &= -\frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\alpha}F^\nu{}_\alpha. \end{aligned} \quad (12)$$

This tensor is evidently different from the canonical stress tensor for the Maxwell field (11). In the following exercise you will prove that this symmetric tensor, $T^{\mu\nu}$ is a "new and improved" stress tensor for Maxwell fields i.e. it is conserved and it gives us right expression fro energy density and momentum density of EM field.

Homework:

1. Check that $T^{\mu\nu}$ is conserved i.e. it is diveregenceless $\partial_\mu T^{\mu\nu} = 0$.

2. Check that indeed components of these coincide with energy density and momentum density of the Maxwell field, i.e. , $T^{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$ and $T^{0i} = \mathbf{E}\times\mathbf{B}$.

3. Check that $T^{\mu\nu}$ is traceless, i.e. $T^\mu_\mu = 0$. [2]

[2] This is a a result of scale invariance of the Maxwell action. A scale transformation is given by scaling all fields and coordinates by their mass (energy) dimensions $x' \to \lambda x$, $A_\mu \to \lambda^{-1}A_\mu$. The scalar action is not invariant under this scaling symmetry due to the "mass term" $\frac{1}{2}m^2\phi^2$ which does not scale properly because m is a parameter and not a field.